Available online at www.sciencedirect.com
ScienceDirect
Economics Letters 98 (2008) 2–8
economics letters
www.elsevier.com/locate/econbase

A note on national income in a dynamic economy

Geoffrey Heal [a,*], Bengt Kriström [b]

[a] *Columbia Business School, United States*
[b] *Department of Forest Economics, SLU, 901 83 Umeå, Sweden*

Received 15 May 2006; received in revised form 25 October 2006; accepted 12 December 2006
Available online 24 September 2007

Abstract

We review the properties of the static measure of changes in economic welfare, national income, in a dynamic economy. It is possible to establish attractive welfare properties by rather simple arguments, and this can be estimated from current data. Its welfare properties are preferable to those of net national product and comparable to those of the state valuation function.

Keywords: Representative agent; National income; Wealth; Sustainability; Cost–benefit

JEL classification: C 610; D 610; D 900; Q 560

1. Introduction

In conventional models of resource allocation the usual measure of economic welfare is national income, which is measured at a competitive equilibrium by the value of output at equilibrium prices. This measure has the extremely attractive property that a small change in resource allocation is a potential Pareto improvement if and only if it leads to an increase in the value of national income. So national income is closely linked to economic welfare and has fundamental welfare significance. Strangely, in the very extensive literature on welfare in dynamic economies, this

* Corresponding author.
E-mail addresses: gmh1@columbia.edu (G. Heal), Bengt.Kristrom@sekon.slu.se (B. Kriström).

0165-1765/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.econlet.2006.12.003

measure has almost never been considered as a welfare measure. Weitzman (1976) famously suggested the Hamiltonian corresponding to a dynamic optimization as a welfare measure in a dynamic economy, but this has limitations, in particular the lack of a close connection with economic welfare, as others have noted and we will note below. Other authors have suggested a linearized version of the Hamiltonian as a dynamic welfare measure (Dasgupta et al., 1995 amongst them). More recently the emphasis has been on wealth as a dynamic welfare measure, consistent with Samuelson's striking intuition that a "wealth-like" measure is called for in measuring welfare in a dynamic context.[1] The present value of consumption along an optimal path is the intertemporal analog of the usual measure and is also, to use Samuelson's terminology, a wealth-like measure, and indeed was suggested by Samuelson as an appropriate measure of 'social income' in a dynamic context.[2] It is a direct extension to the dynamic context of the usual welfare measure, national income, for the static case.[3] In this paper we investigate the properties of this analog to the static measure as a welfare measure in a dynamic economy, and suggest that it has all the attractive properties that it posseses in the static case and is the best measure for both static and dynamic contexts.

There has seemed to be a compelling technical reason for not seeking the equivalent to the usual measure in the intertemporal context, which is that all of the fundamental static welfare properties are established by using separating hyperplane theorems, which are straightforward infinite dimensional commodity spaces but become technically extremely challenging when one considers infinite time periods and hence infinite dimensional commodity spaces (for an attempt see Heal, 1998). In this paper we show that in fact it is unnecessary to use infinite dimensional analysis to derive results for the intertemporal equivalent of national income: all the key results can be derived from dynamic optimization techniques. In addition to the technical challenges, there has been another reason for being reluctant to work with national income in a dynamic context. National income in this context is the value of present and future output at supporting prices, and so appears to require knowledge of future prices for its measurement. We show that in a perfect set of markets, all the information we need about the future is contained in the spot prices of capital goods, so that in fact the national income of an economy that will continue into the future can be measured from currently-observable variables. In effect we rehabilitate national income as a serious welfare measure in a dynamic context. The value of output at supporting prices has exactly the properties we need of a welfare measure and is observable. It is reassuring to have a dynamic measure that is equivalent to the usual and very satisfactory static one.

1.1. The main result

Consider the standard optimal growth or representative agent problem where the consumer has instantaneous utility $u(\boldsymbol{c}_t)$ where $\boldsymbol{c}_t$ is an n-vector of consumption at time t, with typical coordinate i, the production process is given by $\boldsymbol{y}=f(\boldsymbol{s}_t)$ where $\boldsymbol{y}$ is an n-vector of outputs and $\boldsymbol{s}_t$ the n-vector of

[1] Samuelson (1961), Dasgupta and Mäler (2000), Heal (1998), Heal and Kriström (2006), World Bank (2006).

[2] It was in fact called wealth much earlier by Irving Fisher (see Samuelson, 1961).

[3] It also features in the work of Sefton and Weale (2006), who show that it is equal to their definition of real NNP, which differs from several Hamiltonian-based definitions used in the literature.

capital stocks at time t. The national income identity is therefore $\boldsymbol{c}_t=f(\boldsymbol{s}_t)-\dot{\boldsymbol{s}}_t$, with the initial stock of capital given at $\boldsymbol{s}_0$. The intertemporal optimization problem is

$$\max \int_0^\infty u(\boldsymbol{c}_t)e^{-\delta t}\mathrm{d}t \text{ subject to } \boldsymbol{c}_t = f(\boldsymbol{s}_t)-\dot{\boldsymbol{s}}_t,\ \boldsymbol{s}_0 \text{ given} \tag{1}$$

The solution to Eq. (1) is found from the Hamiltonian

$$H = u(\boldsymbol{c}_t)e^{-\delta t} + \boldsymbol{\lambda} e^{-\delta t}[f(\boldsymbol{s})-\boldsymbol{c}] \tag{2}$$

where $\boldsymbol{\lambda}$ is the vector of shadow prices on the capital stocks, with first order conditions

$$u_i' = \lambda_i \textit{ and } \delta - \frac{\dot{\lambda}_i}{\lambda_i} = f_i'$$

where a prime denotes a derivative. If we define national income as the value of consumption at supporting prices, the conventional definition in the finite horizon case, then in this case it is given by the integral

$$\mathrm{NI} = \int_0^\infty \boldsymbol{c}_t \boldsymbol{\lambda}_t e^{-\delta t}\mathrm{d}t$$

Here $\boldsymbol{c}_t\boldsymbol{\lambda}_t$ is the inner product of the consumption and price vectors. As we mentioned, taking this as a definition of national income appears to be problematic from an operational perspective as it contains data about the distant future. But in fact we are really only interested in changes in this value in response to possible policy choices, and not in its absolute value, and it is possible to calculate changes from currently observable data according to the following formulae:

$$\frac{\mathrm{dNI}}{\mathrm{d}\boldsymbol{s}_i} = \lambda_i \frac{\mathrm{dNI}}{\mathrm{d}t} = \boldsymbol{\lambda}\dot{\boldsymbol{s}}$$

where again $\boldsymbol{\lambda}\dot{\boldsymbol{s}}$ is the inner product of two vectors. So the response of NI to a change in the capital stock is given by the price of capital, and the rate of change of NI is the value of investment at current prices. All of these variables are observable. This means that everything we need to know about NI we can observe from capital stocks, their price and their rates of change.

Let V be the state valuation function associated with problem (1), i.e.

$$V = \max \int_0^\infty u(\boldsymbol{c}_t)e^{-\delta t}\mathrm{d}t \quad \text{subject to } \boldsymbol{c}_t = f(\boldsymbol{s}_t)-\dot{\boldsymbol{s}}_t, \boldsymbol{s}_0 \text{ given} \tag{3}$$

Then we can prove the following propositions, which confirm the statements made above and establish that changes in NI are accurate local measure of welfare changes.

Proposition 1. *The vector of derivatives of NI with respect to the stocks equals* $\boldsymbol{\lambda}$ *and the rate of change of NI at time zero is* $\boldsymbol{\lambda}_0\dot{\boldsymbol{s}}_0$.

Proof. We know that $\frac{\partial V}{\partial s_{i,0}} = \lambda_i$. Let $\frac{\partial c_t^*}{\partial s_{i,0}}$ be the derivative of the optimal consumption level at date t with respect to a change in $s_{i,0}$, and let u_t^* be an abbreviation for $u_i'(c_t^*)$. So

$$\lambda_i = \frac{\partial V}{\partial s_{i,0}} = \int_0^\infty \left[\sum_j u_j'(c_t^*)\frac{\partial c_{j,t}^*}{\partial s_{i,0}}\right] e^{-\delta t}\mathrm{d}t \tag{4}$$

$$= \int_0^\infty \left[\sum_j \lambda_{j,t}\frac{\partial c_{j,t}^*}{\partial s_{i,0}}\right] e^{-\delta t}\mathrm{d}t \tag{5}$$

$$= \frac{\partial}{\partial s_{i,0}}\left\{\int_0^\infty \boldsymbol{\lambda c} e^{-\delta t}\mathrm{d}t\right\} = \frac{\partial \mathrm{NI}}{\partial s_{i,0}} \tag{6}$$

We are assuming here that the optimal policy is a smooth function of the initial stocks.[4] This last expression is the derivative of NI with respect to the initial stock, as asserted in the theorem. From this we can establish the result on the time rate of change:

$$\left(\frac{\mathrm{dNI}}{\mathrm{d}t}\right)_{t=0} = \sum_i \frac{\partial \mathrm{NI}}{\partial s_{i,0}}\frac{\mathrm{d}s_{i,0}}{\mathrm{d}t} = \boldsymbol{\lambda}_0\dot{\boldsymbol{s}}_0 \tag{7}$$

□

Proposition 2. *The derivatives of NI with respect to stocks and its rate of change at time zero are identical to those of the state valuation function* (3)*. Hence these changes in NI are accurate measures of welfare changes.*

Proof. Immediate. □

1.2. Illustration

Next we illustrate the results of the previous section in the context of the Hotelling model, the simplest and one of the oldest dynamic welfare models. We compute national income NI for this model and show that $\frac{\mathrm{dNI}}{\mathrm{d}s_0} = \frac{\mathrm{d}V}{\mathrm{d}s_0}$. The basic model is a special case of the representative agent model with a single good: the resource allocation problem is to maximize $\int_0^\infty u(\boldsymbol{c}_t)e^{-\delta t}\mathrm{d}t$ subject to $\int_0^\infty \boldsymbol{c}_t\mathrm{d}t \le \boldsymbol{s}_0$. For this model $\mathrm{NI} = \boldsymbol{\lambda}_0\boldsymbol{s}_0$. The proof is straightforward: on an optimal path $(\mathrm{d}u_c/\mathrm{d}t)(1/u_c) = \delta$ so that $\int_0^\infty \boldsymbol{c}_t u_c e^{-\delta t}\mathrm{d}t = u_c(0)\int_0^\infty c_t\mathrm{d}t = u_c(0)\,s_0$ (see Heal and Kriström, 2006). Hence $\frac{\mathrm{dNI}}{\mathrm{d}s_0} = \lambda_0$. But we know that $\frac{\mathrm{d}V}{\mathrm{d}s_0} = \lambda_0$ so that as the proposition asserts we have

$$\frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{s}_0} = \boldsymbol{\lambda} = \frac{\mathrm{dNI}}{\mathrm{d}\boldsymbol{s}_0} \tag{8}$$

[4] The first order conditions for optimality form a resource allocation mechanism in the terminology of Dasgupta and Mäler (2000), who also assume smoothless.

2. Net national product

An alternative that has been proposed as a measure of wellbeing is net national product or NNP. It has some interesting and useful properties. Principal amongst these is Weitzman's result that the Hamiltonian, which can be interpreted as a measure of NNP in the case of a linear utility function, represents a utility level that if maintained forever would have the same present value as the optimal path, i.e. as the state valuation function. In addition one can show that the time derivative of NNP along an optimal path is the same as that of the state valuation function under certain conditions, more on this below. However, for arbitrary perturbations of the economy the sign of the change in NNP may be different from that of the change in the state valuation function, so that for general changes in the economy NNP is not a satisfactory welfare measure.

The relationship between the time derivatives of NNP and SVF, and is easily established following Heal and Kriström (2006). For simplicity we consider the Ramsey model with a single capital good so that for until further notice all variables are scalars: then (following Eqs. (13) (14) (15) of Heal and Kriström)

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \boldsymbol{\lambda}\dot{\boldsymbol{c}} + \dot{\boldsymbol{\lambda}}\dot{\boldsymbol{s}} + \boldsymbol{\lambda}\ddot{\boldsymbol{s}} = \delta\frac{\mathrm{d}V}{\mathrm{d}t} \text{ and NNP} = \boldsymbol{c} + \dot{\boldsymbol{s}} \tag{9}$$

which together imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{NNP} = \frac{\mathrm{d}V}{\mathrm{d}t}\frac{\delta}{\boldsymbol{\lambda}} - \frac{\dot{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}}\dot{\boldsymbol{s}} \tag{10}$$

Now $\frac{\mathrm{d}V}{\mathrm{d}t} = \boldsymbol{\lambda}\dot{s}$ and of course $\boldsymbol{\lambda}>0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{NNP} = \dot{\boldsymbol{s}}\left(\delta - \frac{\dot{\boldsymbol{\lambda}}}{\boldsymbol{\lambda}}\right) \tag{11}$$

Putting these together we see that $\frac{\mathrm{d}V}{\mathrm{d}t}$ is positive when $\dot{s}>0$ and vice versa and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{NNP}$ also has the sign of $\dot{\boldsymbol{s}}$ if and only if the real return $(\delta - \frac{\dot{\lambda}}{\lambda})$ is positive. This result is stated e.g. in Li and Löfgren (2006).

Asheim and Weitzman (2001) have a more general result. They study the relationship between the movements of NNP and the state valuation function over time, and show that in the one good case with the real interest rate positive the time derivative of NNP has the same sign as that of the state valuation function, and that the properties of the one good case can be recovered in the many-good case by aggregation and the use of a consumer price index, provided that the real interest rate is positive.

To show the limitations of NNP as a general measure of welfare changes, we return to the many-good case where consumption, stocks and prices are vectors and consider a small perturbation of the economy's optimal path by changes in the time paths of consumption and of capital stocks, and compute the resulting changes in the Hamiltonian (Eq. (2)) and in the state valuation function. By using the first order conditions and a linear approximation to the Hamiltonian one can easily show that

$$\Delta H = \sum_i \Delta \boldsymbol{s}_i \boldsymbol{\lambda}_i \left\{\delta - \frac{\dot{\boldsymbol{\lambda}}_i}{\boldsymbol{\lambda}_i}\right\} \tag{12}$$

We know that

$$\Delta V=\sum_{i=1}^{n}\Delta \boldsymbol{s}_i\boldsymbol{\lambda}_i \tag{13}$$

Comparing Eqs. (12) and (13) it is clear that these need not have the same signs. Indeed they can differ in sign even if all real rates of return are positive, if some of the changes in stocks are negative.[5] While we have worked here with the impact of a perturbation on the Hamiltonian, the result would be the same for its impact on linearizations of the Hamiltonian, which is what matters for the first order impact of a small change. Note that we are here dealing with a perturbation of the economy about an optimal path — with evaluating a potential project through cost–benefit analysis.[6] We can summarize this by saying, as has already been noted in the literature (for example Johansson and Löfgren, 1996; Asheim, 2000), that NNP is not an appropriate criterion for cost–beneffit evaluations, except under special conditions. Asheim (2006) provides a comprehensive recent review of the welfare properties of NNP.

The conclusion here is that the value of changes in NI, and also the value of changes in capital stocks, are good indicators of welfare changes for both perturbations about a path and movement along an optimal path, whereas NNP works only for changes along a path, and provided the right aggregation procedure is used and the real interest rate is positive.

3. Measuring sustainability

An interesting aspect of these results is that $\boldsymbol{\lambda}\dot{\boldsymbol{s}}$ is observable in principle, depending only on current values of variables, and indeed is what Hamilton and Clemens have termed "genuine savings". (See Hamilton and Clemens, 1999; World Bank, 2006 and also Arrow et al., 2004). Hamilton and Clemens suggest that this is a measure of sustainability. So the change in NI is observable and equals a measure that has been suggested as appropriate for measuring how an economy's long-term productive potential is changing, how sustainable is its development. This is an insight that Samuelson (1961) missed — he states that we need a "wealth-like measure" and that "I know of no way of even approximating from market valuations of factors" what this measure would be.

There is another direct connection to the literature on sustainability, as the state valuation function measures the economy's long-term welfare potential, and that this be non-decreasing has been suggested as an interpretation of sustainability (Heal, 1998; Arrow et al., 2004). In this context sustainability is equivalent to non-decreasing national wealth (Dasgupta and Mäler, 2000; Dasgupta, 2001 amongst others), and the change in national income as measured here is the same as the change in the state valuation function.

References

Arrow, K.J., Dasgupta, P.S., Goulder, L., Daily, G.C., Ehrlich, P.R., Heal, G.M., Levin, S., Maler, K.-G., Schneider, S., Starrett, D., Walker, B., 2004. Are we consuming too much? Journal of Economic Perspectives (Summer).

Asheim, G.B., 2000. Green national accounting: why and how? Environment and Development Economics 5, 25–48.

[5] Of course if there is only one good and the real return is positive then the two do have the same sign.

[6] If we look at the evolution of the Hamiltonian and of the state valuation function along an optimal path matters are different: from the fact that at all times $H=\delta V$ it of course follows that $\mathrm{d}H/\mathrm{d}t=\delta \mathrm{d}V/\mathrm{d}t$ and the signs of the time trends in H and SVF are the same.

Asheim, G.B. Can NNP be used for welfare comparisons? Mimeo, February 21 2006. Available at http://folk.uio.no/gasheim/welcom02.pdf.

Asheim, G.B., Weitzman, M.L., 2001. Does NNP growth indicate welfare improvement? Economics Letters 73, 233–239.

Dasgupta, P.S., 2001. Valuing objects and evaluating policies in imperfect economies. Economic Journal 111 (471), May, 1–29.

Dasgupta, P.S., Mäler, K.-G., 2000. Net national product, wealth and social well-being. Environment and Development Economics 5, 69–93.

Dasgupta, P.S., Kriström, B., Mäler, K.-G., 1995. Current issues in resource accounting. In: Johansson, P.-O. (Ed.), Current Issues in Environmental Economics.

Hamilton, K., Clemens, M., 1999. Genuine savings rates in developing countries. World Bank Economic Review 13 (2), 333–356.

Heal, G.M., 1998. Valuing the Future: Economic Theory and Sustainability. Columbia University Press.

Heal, G.M., Kriström, B., 2006. National income and the environment. In: Mäler, K.-G., Vincent, J. (Eds.), Handbook of Environmental Economics, North-Holland, Amsterdam.

Johansson, P.-O., Löfgren, K.-G., 1996. On the interpretation of 'Green' NNP measures as cost–beneffit rules. Environmental and Resource Economics 7, 243–250.

Li, Z., Löfgren, K.-G., 2006. Comprehensive NNP, social welfare, and the rate of return. Economics Letters 90, 2254–2259.

Samuelson, P.A., 1961. The evaluation of 'social income', capital formation and wealth, chapter 3, 32–57. In: Lutz, F.A., Hague, D.C. (Eds.), The Theory of Capital. St. Martins Press, New York.

Sefton, J., Weale, M.R., 2006. The concept of income in general equilibrium. Review of Economic Studies 73 (1), 219–249 (January).

Weitzman, M.L., 1976. On the welfare significance of net national product in a dynamic economy. Quarterly Journal of Economics 90 (1), 156–162.

World Bank, 2006. Where is the wealth of nations? Measuring Capital for the 21st Century. The World Bank, Washington, D.C.