Social choice and resource allocation: a topological perspective

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1. Topological social choice

This volume brings together a collection of papers in an area which has evolved rapidly in recent years. This is the area of continuous or topological approaches to the study of social choice, and the analysis of what have come to be known as "Chichilnisky rules". Initiated by Chichilnisky in a paper in *Advances in Mathematics* in 1980 [14], the field has quickly evolved from a separate and distinct area of social choice to one which is integrated into the mainstreams of both social choice and the general equilibrium approach to markets, and which in turn has come to integrate and bridge these previously disparate areas, revealing deep and unexpected connections between them. Many of the papers in this volume were first presented either at a conference on "*Ethics, Economics and Business*" held at Columbia Business School in March 1993, or at a workshop on "*Geometry, Topology and Markets*" held at the Fields Institute for Mathematical Sciences in July 1994. These two meetings were important in catalyzing the growth of this area. My aim in this introduction is to place the papers in this volume in the context of the development of the area in general, and then to give some personal reflections and observations on the growth of this area, its future, and its role in economic theory. Three of the sections of this introduction, "Topological Social Choice", "Incentives", "Infinite Populations", match the three sections into which the papers that follow have been organized. Section 2 of this introduction, on "Social Choice and Resource Allocation", surveys recent results on the relationship between social choice theory and resource allocation theory, a field covered in more detail in Chichilnisky's paper [18] in Sect. 1. This is such a new and fascinating area that it seems to merit specific comments. The final section contains comments and reflections on the contribution made by this area, and on its future role.

1.1. Chichilnisky's formulation

Chichilnisky's original reformulation of the social choice problem differed from Arrow's [1] in one crucial and strategic respect. Rather than working with a discrete choice space, and treating preferences as orderings over this, i.e., as rankings of a discrete set of objects, she made an assumption conventional in the rest of economic theory and took the choice space to be Euclidean, with preferences given by the indifference surfaces familiar from consumer theory. This immediately placed her version of the social choice problem close to classical resource allocation theory, in that she worked with the same categories of objects. It also enabled her to focus on the problem a set of mathematical tools that are more powerful than those of discrete mathematics, which had previously characterized research methods used in social choice. Within this framework she asked the following question:

> When is there a social choice rule (a map from preference profiles to preferences) which is continuous, anonymous, and which respects unanimity?

Continuity is a conventional axiom in resource allocation theory, and indeed in most applications of mathematics, but is nevertheless a different type of axiom from those used by Arrow and his followers, although Baryshnikov in this volume suggests that Arrow's axiom of independence of irrelevant alternatives plays a similar role and in fact implies continuity once the Arrow problem is embedded in a more structured framework. However, anonymity and respect of unanimity are of the same type as used previously, and are relatively weak. Anonymity is an "equal treatment" axiom: what matters is not who votes, but which way they vote. In particular it implies the widely-used non-dictatorship axiom. Respect of unanimity is a very weak condition and simply means that if everyone in society has the same overall preference ordering, then society adopts this common ordering as its own. This is implied by, but does not imply, the Pareto axiom, according to which if all agents prefer alternative A to alternative B, then society must rank A above B.

Within this framework, let the space of preferences on a Euclidean choice space be $\mathscr{P}$, and the number of agents be k. Then one is asking for a map Φ from $\mathscr{P}^k$ to $\mathscr{P}$ which is continuous, the identity on the diagonal[1] $D = \{p_1, \ldots, p_k \colon p_i = p_j \; \forall i,j\}$ and invariant under permutations of its arguments[2], so that $\Phi\ (p_1 p_2, \ldots, p_\pi) = \Phi(p_{\pi 1}, p_{\pi 2}, \ldots, p_{\pi k})$ where $\pi 1, \ldots, \pi k$ is a permutation of the integers 1 to k.

This is an elegant framework admitting analysis by topological methods, which show that in general there is no such rule. This is *Chichilnisky's impossibility theorem*: "*There exists no social choice rule Φ from $\mathscr{P}^k$* to $\mathscr{P}$ *which is continuous, anonymous and which respects unanimity*". In fact her argument in [14] is that if there were such a rule Φ, then there would in particular be a map ϕ from $\mathscr{S}^k$ to $\mathscr{S}$ which has the same properties, where $\mathscr{S}$ is the $n-1$-dimensional sphere in the choice space $\Re^n$. Chichilnisky shows that there is no such map on spheres, and hence there is no social choice rule of the

[1] This is respect of unanimity.

[2] This is anonymity.

desired type. This framework, and the mathematical arguments and concepts which support it, are surveyed in the paper by Mehta [33] in this volume. This paper by Saari [36] also reviews the structure of social choice theory from a topological and geometric perspective.

This simple and intuitive framework leads to deep results in the area of social choice, and also, more surprisingly, to the development of a very fundamental connection between social choice theory, and the theory of competitive equilibria and cores in exchange economies.

1.2. The choice of topology

A topic on which there has been some confusion, is the role played by the choice of a topology on the space of preferences in Chichilnisky's work. Her basic impossibility theorem is: "*There exists no social choice rule Φ from $\mathscr{P}^k$ to $\mathscr{P}$ which is continuous, anonymous and which respects unanimity*". In her original proof [14], the only condition needed on the topology on the space of preferences for the validity of this result, is that the space of linear preferences on the Euclidean choice space should inherit from this topology the "usual" Euclidean topology. The point here is that linear preferences are a particularly simple type of preference: they can be represented by unit vectors given by their normalized gradient vectors, and so have a natural topology, the Euclidean topology on vectors in finite dimensional spaces. The set of linear preferences can thus be identified with the set of unit vectors in the choice space $\Re^n$, and so with the unit sphere $\mathscr{S}^{n-1}$ of dimension $(n-1)$. Chichilnisky's original argument works with any topology on the overall preference space which reduces to this topology when one considers the subspace of linear preferences. Any reasonable topology, certainly any topology which has been used on preference spaces, satisfies this condition[3].

[3] It is satisfied, for example, both by the smooth topology and also by the closed convergence topology. There is a relatively straightforward explanation of the role played by the condition that the space of linear preferences should inherit from the topology on the overall preference space the "usual" Euclidean topology. We seek a continuous map $\Phi\colon \mathscr{P}^k \to \mathscr{P}$ which is the identity on the diagonal of $\mathscr{P}^k$, and invariant under permutations of its arguments. Consider the space $\mathscr{L}$ of linear preferences on the choice space. Clearly $\mathscr{L}$ is a subspace of $\mathscr{P}$, and the required map Φ would induce a map with the same properties from $\mathscr{L}^k \to \mathscr{P}$. Let $\mathscr{I}$ be the map which includes $\mathscr{L}$ in $\mathscr{P}$. So the composition $\mathscr{I} \circ \Phi$ maps from $\mathscr{L}^k \to \mathscr{P}$. Now consider the map which associates any social preference with its gradient vector at a point, any point, in the choice space. This is the evaluation map: denote it $\mathscr{E}$. This maps social preferences into linear preferences on the choice space. So the overall composition $\mathscr{I} \circ \Phi \circ \mathscr{E}$ sends k-tuples of linear preferences into a linear preference. In this operation, the k-tuple of linear preferences is an element of the overall preference space. If the subspace of linear preferences has the usual Euclidean topology, one has constructed a continuous map from the k-fold product of spheres to the sphere which is the identity on the diagonal of $\mathscr{P}^k$, and invariant under permutations of its arguments. But by analysis of the homotopy groups of these spaces Chichilnisky shows this to be impossible: hence her theorem.

Subsequently it has proved possible to dispense with even this limitation on the topology. Chichilnisky and Heal [21] establish that contractibility[4] of the space of preferences is both necessary and sufficient for the existence of Chichilnisky rules. The Chichilnisky–Heal theorem is as follows: "*Let $\mathscr{P}$ be a para-finite CW complex*[5]. *Then there exists a function Φ from $\mathscr{P}^k$ to $\mathscr{P}$ which is continuous, anonymous and respects unanimity if and only if $\mathscr{P}$ is contractible*". This result implies Chichilnisky's 1980 result, and is true whatever the topology on the preference space, provided that continuity of the social choice rule and continuity in defining contractibility is with respect to the topology on the space of preferences. In this sense it is analogous to statements with which economists are more familiar such as fixed point theorems: "*A continuous function from a convex compact topological space into itself has a fixed point*". This statement is independent of the choice of topology: it is true for any function which is continuous in a topology in which the space is compact, whatever that topology. In fact there is an important mathematical connection between Chichilnisky's impossibility theorem and fixed point theorems: in certain cases her impossibility theorem is equivalent to Brower's fixed point theorem [13, 16]. Related points are made in the paper by Zhou [39] in this volume.

2. Social choice and general equilibrium

One of the most striking results to emerge from topological social choice is the equivalence between Arrow's paradox, Chichilnisky's paradox, the existence of a competitive equilibrium in a general equilibrium exchange economy, and the non-emptiness of the core. This is a remarkable set of relationships:

Arrow Paradox $\Leftrightarrow$ Chichilnisky Paradox $\Leftrightarrow$

Competitive equilibrium $\Leftrightarrow$ Core

This shows that the problem of social choice is an integral part of *the* problem in economics: that rather than being a non-economic problem, as asserted by Samuelson [38] in his review of Arrow's book, all forms of resource allocation (via cooperative games, competitive exchange economies and social choices) are equivalent, at least in terms of the conditions under which they yield interesting outcomes. Social choice has come in from the cold, and can contribute to the resolution of the most classical questions in economics.

This result depends on three pairwise equivalences:

1. the equivalence of the existence of a competitive equilibrium in an exchange economy and the existence of social choice rules in Chichilnisky's sense,

[4] A topological space Y is contractible if $\exists y_0 \in Y$, and a continuous function $f: Y \times [0, 1] \to Y$, with $f(y, 0) = y \; \forall y \in Y$ and $f(y, 1) = y_0 \; \forall y \in Y$: see Heal [24] and also the paper by Mehta in this volume.

[5] This is a CW complex with a finite number of cells in each dimension (though not necessarily a finite number of cells in total). See also the paper by Baryshnikov in this volume.

2. the equivalence of the existence of a competitive equilibrium and the non-emptiness of the core, and
3. the equivalence of Chichilnisky's formulation of the social choice paradox[6] and Arrow's original formulation[7].

The first two equivalences are reviewed here in the paper by Chichilnisky [18], and the latter in the paper by Baryshnikov [4] and also for certain cases in the paper by Chichilnisky.

The equivalence of existence of a Chichilnisky rule to the existence of a competitive equilibrium was shown by Chichilnisky in 1991 [12], where she established that a condition called "limited arbitrage" was necessary and sufficient both for the existence of a competitive equilibrium and for the existence of a social choice rule in her sense. Limited arbitrage is a condition requiring similarity of agents' preferences and endowments. Loosely speaking, agents' preferences and endowments satisfy limited arbitrage when the directions in which their utilities increase without bound are similar. Precisely, the condition is the that duals of these sets of directions have a non-empty intersection[8].

In her paper [16], Chichilnisky shows that for very general families of sets, their intersection is non-empty if and only if their union is contractible. In so doing she uses the concept of the nerve of a family of sets, also used by Baryshnikov in this volume. This result enables her to establish that the similarity of preferences which is necessary and sufficient for the existence of a competitive equilibrium, and which is stated in terms of non-empty intersection of a family of sets associated with agents' preferences, is the same as the contractibility condition on the space of preferences shown by Chichilnisky and Heal [21] to be both necessary and sufficient for the existence of Chichilnisky rules. The Chichilnisky–Heal result provided for the first time a clear and complete characterization of the conditions for a resolution of the social choice paradox.

In 1993 Chichilnisky [10] showed that the same conditions are necessary and sufficient for the non-emptiness of the core of an exchange economy: this is an almost immediate corollary of her earlier results.

There are two separate results on the relationship between Arrow's and Chichilnisky's axiomatizations. One is due to Chichilnisky [9, 18]: she shows that for choices over sets of alternatives involving large utility values, the limited arbitrage condition is equivalent to the non-existence of Condorcet triples. From this she can show that limited arbitrage is also equivalent to the non-existence of Arrow's paradox over choices between alternatives involving large utility values. The concept of "choices involving large utility values" is a novel one: Chichilnisky uses this to refer to a property (such as the presence or absence of Condorcet triples) which holds on sets of alternatives whose utility values exceed some lower bound (which of course is less than the supremum of utility values). The focus on large utility values arises because

[6] This was in terms of rules which are continuous, anonymous and respect unanimity.

[7] This was in terms of rules which work on a universal domain, and satisfy the Pareto principle, independence of irrelevant alternatives and non-dictatorship.

[8] Limited arbitrage bounds, but not by zero, the gains in utility available from zero cost trades.

limited arbitrage, which was shown by Chichilnisky to be necessary and sufficient for the existence of competitive equilibrium and for the non-emptiness of the core, deals with the asymptotic behavior of preferences.

Baryshnikov completed the picture in 1993 [3] by showing Arrow's formulation of social choice theory to have the same underlying structure as Chichilnisky's: he showed that Arrow's axioms imply the existence of a map from (at the topological level) the product of spheres to the sphere. Recall that Chichilnisky's original argument proceeded by the analysis of a map from the product of spheres to a sphere, and that Chichilnisky's impossibility theorem implies that there is no such map: hence Chichilnisky's result implies Arrow's. Arrow's assumption of independence of irrelevant alternatives ensures that this is a simplicial map and so continuous. The Pareto condition used by Arrow implies respect of unanimity. Baryshnikov's conclusion is that the existence of an Arrow map is fundamentally equivalent to the existence of a Chichilnisky map: both are tied to the existence of symmetric maps on spheres.

So, in summary, Chichilnisky has shown (via a very simple proof) that her formulation and Arrow's are equivalent for choices involving large utility values. Baryshnikov has shown that Arrow's formulation can be reduced to the study of maps from the product of spheres to a sphere, and has the same structure as Chichilnisky's.

The full implications of this set of results are clearly not yet understood. One aspect in particular may lead to interesting new developments: this is the introduction of a formal measure of "social diversity" in Chichilnisky's proofs of equivalence [9, 11]. She shows that competitive equilibria, the core and social choice rules all exist if and only if a certain index of social diversity takes the value one, its minimal value. This index assuming the value unity is equivalent to the limited arbitrage condition. Non-existence of any of these equilibrium concepts can therefore be attributed to the degree of social diversity. This is quite in keeping with the early literature on social choice theory which, from the time of Black [6] onwards, has introduced limitations on the diversity of preferences[9] as a way of ensuring the existence of social choice rules. What is elegant, unexpected and far-reaching in its implications, is that the existence of competitive equilibria and the core also depend on this same concept of diversity.

An important feature of topological social choice is the natural and productive role that it has provided for interesting[10] mathematics in economics: Chichilnisky's original formulation of the problem, Baryshnikov's proof of equivalence between Chichilnisky's and Arrow's formulations, and Chichilnisky's proofs of the equivalences listed above [16], all require the use of techniques from algebraic topology not previously used in economics, but which nevertheless seem well-suited to our subject. The techniques of topology are designed for analysis of problems which are qualitative rather than quantitative in nature, either because numerical representation is not possible or because it is possible only subject to considerable error. They therefore fit

[9] Such as single peakedness (Black) and value restrictedness (Pattanaik and Sen [34]).
[10] Some readers may be inclined to remind me of the Chinese curse "May you live in interesting times!".

well the areas of welfare economics and social choice, and it is encouraging that they have yielded a rich harvest.

3. Incentives

3.1. *Manipulability and straightforwardness*

One of the most productive developments of social choice theory has been in the area of incentive compatibility and the manipulability of social choice procedures. Gibbard [28] and Satterthwaite [37] observed that the non-existence of social choice rules implied that most social choice rules could be manipulated, in the sense that agents' optimal strategies typically involved revealing preferences other than their true preferences. This opened a productive literature on incentive compatibility, addressing this both in the strict confines of the traditional social choice framework, and in the more economic confines of the selection of levels of provision for public goods, the culmination of which was the Groves–Ledyard mechanism [27] for determining efficient allocations of resources to the provision of public goods and overcoming the classical free rider problem. A substantial part of this literature has been concerned to characterize social choice rules which are "straightforward", in the sense that for all agents facing these rules, revealing their true preferences is a dominant strategy so that they have no incentive to behave strategically or dishonestly. The paper by Chichilnisky and Heal in this volume [22], often cited but not previously published, gives a complete characterization of straightforward rules for a reasonably broad class of preferences, those which are single peaked. It shows that such rules have a remarkably simple structure: they are locally either dictatorial or constant. Clearly rules which are everywhere constant, or everywhere dictatorial, are straightforward: no-one has any incentive to misrepresent. What is surprising, is that all straightforward rules are obtained by "patching together" such functions. The paper by Rasmussen [35] links the literature on implementation to that on topological social choice.

3.2. *Strategic dictators*

In the topological framework there is a result that is equivalent to that of Gibbard and Satterthwaite: this is that any social choice rules which are continuous and respect unanimity, must admit "strategic dictators" [8]. A strategic dictator is an agent who has the following ability. Whatever the preferences announced by other agents, he or she can always announce a preference such that the outcome of the social choice rule will be his or her true preference. In other words, such an agent always has available a response to any set of moves by the other agents which will produce as the outcome his or her optimal outcome.

A strategic dictator is defined formally as follows. Let $\Psi : \mathscr{P}^k \to \mathscr{P}$ be a continuous social choice rule. Let $s \in \mathscr{P}^k$ be a k-vector of preferences announced by all k agents, and let s_{-i} be the $k-1$ tuple of preferences left when agent i's is removed. Let agent i's true preference be s_i^*. Then agent i is a strategic dictator if for any vector s_{-i} of preferences announced by other agents, there exists a response by agent i, denoted s_i (s_{-i}), such that the outcome is precisely the true preference of i, i.e. $\Psi(s_i(s_{-i}), s_{-i}) = s_i^*$. Typically, of course, $s_i(s_{-i}) \neq s_i^*$, so that agent i obtains the desired outcome by misrepresentation. So in the Chichilnisky framework, manipulability is again implied by the basis impossibility result. (The papers by Baryshnikov [4] and Mehta [33] in this volume also develop this concept.)

The analysis of strategic dictators requires sophisticated mathematical tools: in mathematical terms it has required the classification of maps from the product of spheres to the sphere, a classical mathematical problem. There are two approaches, via homotopy theory and via homology theory. The former involves showing that any continuous social choice rule which satisfies the Pareto condition and an additional condition of positive association[11] must be homotopic to a map which projects the k-tuple $(p_1, p_2, \ldots, p_k)$ onto one of its components, say the jth. Any such projection map is of course dictatorial, with the jth. agent as the dictator. Two maps $f, g: X \to Y$ between the topological spaces X and Y are homotopic if there exists a continuous map $h: X \times [0, 1] \to Y$ such that $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ $\forall x \in X$. In this case the map f can be continuously deformed in to the map g. Chichilnisky [17] formulated the problem of the dictatorial properties of continuous social choice rules in terms of the properties of maps from sphere products to the sphere, and then showed that any continuous map satisfying the Pareto condition and positive association is homotopic to a dictatorial map. In the case of only two agents, the weak association condition is not needed. However, Baryshnikov has shown [5] that in general one cannot dispense with positive association.

An alternative approach is to show that any continuous rule which satisfies only respect of unanimity is homologically equivalent to a projection map and hence to a dictatorial map. Homological equivalence is weaker than homotopic equivalence: if two functions f and g are homologically equivalent, this does not imply that there is a homotopy between them and that one can be deformed into the other. However, this property of being homologically equivalent to a dictatorial rule is still sufficient to establish the existence of a strategic dictator. This is the approach taken in Chichilnisky [8] and in Chichilnisky and Heal [20]: this latter paper considers the case of two agents and in this case the homology argument is equivalent to an argument about the degree of a map from a sphere to a sphere. Chichilnisky shows that any continuous social choice rule which respects unanimity is homologically equivalent to a dictatorial rule and hence that such a rule must be associated with a strategic dictator. If the rule also satisfies the Pareto condition, then the strategic dictator is unique. Koshevoy, in this volume [29], extends Chichilnisky's work on the homotopy equivalence case by dropping the condition of positive association, using only continuity and the Pareto condition, but restricting the choice space to be two dimensional and preferences to be linear. He also extends Chichilnisky's general homological results to the case of infinite populations.

[11] Weak positive association (wpac) in the paper by Koshevoy in this volume.

Chichilnisky and Heal [20] begin to explore the implications of these results for game theory. The key point is as follows: if agent i is a strategic dictator, then at a Nash equilibrium of agents' choices of preferences, the outcome will be agent i's true preference. But if there are two distinct agents who are strategic dictators, then the outcome must equal the true preferences of both. This is a contradiction, implying that in this case there will be no Nash equilibrium to the preference revelation game. This game will have an equilibrium only if there is a unique strategic manipulator, in which case the equilibrium will of course be his or her preferred outcome.

4. Infinite populations

It was recognized by Fishburn [26] and Kirman and Sonderman [25] that moving to an infinite population of voters made a fundamental difference to the social choice problem in Arrow's framework. To be precise, they showed that Arrow's paradox can be resolved in this context. Chichilnisky and Heal, in a paper initially circulated in 1979 but published for the first time in this volume [23], studied the structure of continuous social choice rules with infinite populations. They showed that, as in the discrete case, with infinite populations the resolution is in some sense more apparent than real: both Arrow and Chichilnisky show that "desirable" social choice rules are dictatorial, directly or strategically. This is still true with infinite populations of voters, but the dictators are now "invisible": Chichilnisky and Heal show that they are, for example, the limits of sequences of voters, or generalizations of limits via ultrafilters. In fact they provide a characterization of all continuous rules admissable with infinite populations. They also show that although no individual agent is a dictator, if the set of agents (the integers) is given a finite measure, then there exists a coalition of agents of arbitrarily small measure which determines fully the outcome of the social choice rule.

This approach has recently been applied to intertemporal choice, where there is naturally an infinite sequence of generations. Chichilnisky [7] has used an extension of the topological approach to social choice with infinite populations to provide an axiomatization of alternatives to the utilitarian framework to intertemporal welfare economics, and has related this to the concept of sustainable growth. She uses three axioms, a continuity condition, and then two non-dictatorship conditions, non-dictatorship of "the present" and non-dictatorship of "the future". She shows that these axioms imply that infinite utility streams must be evaluated according to two terms, one which is the integral of the stream against a countably additive measure (such as an integral of discounted utilities) and one of which is a purely finitely additive measure, such as the long run average utility level or the lim sup or lim inf of utility levels. This axiomatization, derived from topological social choice theory, therefore justifies placing more weight on the very long run than the standard discounted utilitarian approach, and also justifies the use of a mix of the intertemporal optimization criteria used in optimal growth theory (discounted integral of utilities) and in the theory of repeated games (long run average utility value).

Lauwers [30, 31, 32] has also analyzed intertemporal welfare economics within this framework, focussing on issues arising from alternative interpretations of the anonymity axiom with infinite populations. In the work of Lauwers, and that of Chichilnisky, a natural and important role emerges for purely finitely additive measures defined on consumption or utility sequences. These purely finitely additive measures correspond to the limits and generalized limits which emerge as "invisible dictators" in the analysis of social choice with infinite populations. Lauwers' work is contained in his papers in this volume.

The paper in this volume by Candeal et al. [19] is also in the tradition of social choice rules with infinite populations, but considers the technically more demanding case of a continuum of agents. In this case the interpretation of axioms like anonymity is particularly delicate, and it is difficult both to formalize this and to establish positive results about the existence of rules satisfying it. Candeal, Chichilnisky and Indurain propose a formalization and show that with this approach one can establish a generalization of the Chichilnisky–Heal possibility theorem to the effect that contractibility of the space of preferences is necessary and sufficient for the existence of continuous rules which satisfy anonymity and respect of unanimity.

5. Where have we come from? Where are we going?

As economists, we are concerned to understand the mechanisms by which resources may be allocated, and to provide a basis for evaluating these. In an increasing number of countries, the market is a predominant mechanism. But even in the most market-oriented economy, many dimensions of resource allocation are within the domain of political decision-making, and are therefore determined by social choices. One thinks immediately of defense, of education, of health care, and of infrastructure investments. These account for probably one third of gross national product in most industrial economies. And even in those areas which are the domain of the market, many collective decisions are made concerning the framework within which the market will operate. Examples are telecommunications and broadcasting. In fact, even in the securities business, one of the most competitive of industries and the archetype of capitalist success, many collective decisions determine the laws which regulate the organization and conduct of the industry. So social choices and social decision-making are integral to the agenda of economics: they are important determinants of the pattern of resource allocation in a society. As economists, we therefore need to understand them, and to understand how they relate to other resource-allocation mechanisms. This is why social choice theory matters.

Prior to Chichilnisky's 1980 article, the paradigm for social choice was still that set by Arrow over fourty years ago. This framework had been productive in exploring the robustness of Arrow's original result with respect to alterations of the axiom set, and had shown the basic paradox to be strong. It had also opened up the area of incentive-compatibility. But it seems fair to say that it had been weak in establishing clear general results and a unified perspective on the subject, and in providing links to the rest of economics. Together with the importance of the underlying subject matter, this provided an opening for an alternative approach.

In the early days of the topological approach to social choice, a comment often heard (though to the best of my knowledge never written down) was that this was not *really* social choice in the traditional sense: it was a different and largely mathematical exercise. We now know that to be wrong: the relationships established between Chichilnisky's and Arrow's formulations show that at a deep level they are capturing the same phenomenon, a fundamental aspect of the process of combining individual preferences into a social preferences through democratic institutions. Furthermore, we also know now that the possibilities and limitations facing us for combining individual preferences through democratic institutions are not very different from those available to us when we look at alternative institutions, such as competitive markets or cooperative games. This naturally leads one to ask whether there are other institutions which are more robust in their domain of applicability, and could provide satisfactory resolutions of social choices (including the allocation of resources) in the face of social diversity greater than that which can be managed by the institutions reviewed in this volume. This seems to be a high priority area for research, although a difficult one. One might say that the results to date have led us to understand better what we know, and to see how various parts of this fit together. Now we need to apply this methodology to problems to which we do not yet have answers, perhaps to problems which we have not yet posed.

There is an important difference in emphasis between the two approaches: the combinatorial approach was always taken to imply that it is very hard to find workable and ethically desirable social choice procedures. Arrow's book, for example, is remembered for the impossibility theorem. The topological approach has made social choice seem more attainable: it has shown that for the existence of social choice rules one needs only that the space of agents' preferences be contractible, a relatively mild limitation on the diversity of individual preferences. This approach has also shown us that it is as easy, or as difficult, to find acceptable social choice rules, as it is to find competitive equilibria in exchange economies, or cores of exchange economies. We are accustomed to thinking of this as easy, although in fact early on in the development of general equilibrium theory Arrow provided a rather robust example of an exchange economy with no competitive equilibrium (repeated in Arrow and Hahn [2] and also in Chichilnisky [11]). Of course, there is really no conflict between these interpretations: it is a matter of what the various authors have chosen to emphasize, and also a product of the fact that the combinatorial approach never produced a complete characterization of the conditions needed for the existence of an acceptable social choice rule, so that we never had a clear picture of how weak these could be, whereas such a characterization was provided relatively early on in the development of Chichilnisky's approach.

Perhaps a fair way of characterizing a part of the difference between the traditional and the topological approaches, is in terms of a trade-off between the investment required to enter, and the long-run payoffs available. The traditional approach, relying on combinatorics and logic, needed relatively little investment in specialized techniques on the part of researchers wishing to work in the area: anyone with a certain degree of analytical sophistication could master the techniques and enter the area. But the ultimate payoffs were limited: the full structure of the social choice problem and its relationship to the rest of resource-allocation theory could not be appreciated by using this methodology. In contrast, algebraic topology has set-up costs: its requires an investment of time before it can be used. Once it is understood, however, it gives greater insights into the structure of the area at relatively low incremental cost. It also gives a geometric insight into the structure of social choice theory, as emphasized in the papers of Baryshnikov and Saari that follow. I was struck by the statement in Baryshnikov's conclusion that he started reproving the Arrow theorem topologically because he had no intuitive feeling for the result from a combinatorial argument, and wanted a proof that matched this striking and beautiful result in aesthetic appeal! This matches well my own motives for becoming involved in this area.

There is a greater fixed cost, but a lower marginal cost, associated with the topological approach: a researcher in the field for the long term should therefore make the investment. In some respects, this is a metaphor for what has happened in much of economic theory: economists have tended to stay within the areas which can be understood by traditional mathematical techniques requiring little investment, and have reached or passed the point of diminishing returns in applying these techniques. An interesting analogy is with the area of dynamical systems: difference equations are relatively easy to formulate and to analyze. They have low set-up costs. But real progress in dynamical systems depended upon the introduction of the continuous approach and the use of a technology based on continuous and differentiable systems.

Reviewing the results in this volume, and their predecessors, one cannot avoid the conclusion that the topological approach has yielded very substantial insights into social choice theory and its relationship to resource allocation theory and mainstream economics, and has done so in a relatively short time. Chichilnisky's first paper appeared in 1980, and given its novelty and the technical demands it placed on the reader, it is natural that it was several years before this was understood and used by others. Since then, however, this approach has generated a large literature and has to its credit an elucidation of the conditions on preference diversity under which social choice is or is not possible, a characterization of the social choice rules which are then possible, showing these to be generalized means, a clarification of the relationship with prior work, and an integration of social choice into the mainstream of economics, leading to surprising and fundamental insights into the relationships between different resource allocation mechanisms. It also has insights, whose potential has not yet been exploited, into the theory of non-cooperative games and into the general area of social diversity, its measurement and its role in the functioning of society. Such achievements and potential constitute a validation of the framework which has produced them. Realizing the potential of this approach in the areas of games and of understanding social diversity seems to offer very high rewards.

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Market arbitrage, social choice and the core

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Abstract. This paper establishes a clear connection between equilibrium theory, game theory and social choice theory by showing that, for a well defined social choice problem, a condition which is necessary and sufficient to solve this problem – limited arbitrage – is the same as the condition which is necessary and sufficient to establish the existence of an equilibrium and the core. The connection is strengthened by establishing that a market allocation, which is in the core, can always be realized as a social allocation, i.e. an allocation which is optimal according to an ordering chosen by a social choice rule. Limited arbitrage characterizes those economies without Condorcet triples, and those for which Arrow's paradox can be resolved on choices of large utility values.

1. Introduction

Markets provide a widely used solution to the problem of allocating resources among the members of the economy. A market equilibrium is individually optimal and clears the markets. The efficiency of competitive market allocations is what makes them desirable.

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